

A Full Multigrid Method For Semilinear Elliptic Equation*

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Abstract

A full multigrid finite element method is proposed for semilinear elliptic equations. The main idea is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. The linearized boundary value problems are solved by some multigrid iterations. Besides the multigrid iteration, all other efficient numerical methods can also serve as the linear solver for solving boundary value problems. The optimality of the computational work is also proved. Compared with the existed multigrid method which need the bounded second order derivatives of the nonlinear term, the proposed method only need the bounded first order derivative of the nonlinear term.

Keywords. semilinear elliptic problem, full multigrid, multilevel correction, finite element method.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

The purpose of this paper is to study the multigrid finite element method for semilinear elliptic problems. As we know, the multigrid and multilevel methods [3, 4, 5, 6, 9, 14, 15, 16, 21] provide optimal order algorithms for solving boundary value problems. The error bounds of the approximate solutions obtained from these efficient numerical algorithms are comparable to the theoretical bounds determined by the finite element discretization. In the past decade years, some researches

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about multigrid method for nonlinear elliptic problem are studied to improve the efficiency of nonlinear elliptic problem solving, i.e. [16, 22, 23]. The Newton iteration is adopted to linearize the nonlinear equation in these existed multigrid methods and then they need the bounded second order derivatives of the nonlinear terms. For more information, please refer to [10, 16, 22] and the references cited therein.

Recently, a type of multigrid method with optimal efficiency for eigenvalue problems has been proposed in [12, 17, 18, 20]. The aim of this paper is to present a full multigrid method for solving semilinear elliptic problems based on the multi-level correction scheme [17, 18]. The main idea is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. For the linearized elliptic problem, it is not necessary to solve the linear boundary value problem exactly in each correction step. Here, we only do some multigrid iteration steps for the linear boundary value problems. In this new version of multigrid method, solving semilinear elliptic problem will not be much more difficult than the multigrid scheme for the corresponding linear boundary value problems. Compared with the existed multigrid method for the semilinear problem, our method only needs the bounded first order derivative of the nonlinear term.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the semilinear elliptic problem. A type of full multigrid method for the semilinear elliptic problem is given in Section 3. In Section 4, some numerical examples are provided to validate the efficiency of the proposed numerical method. Some concluding remarks are given in the last section.

2 Discretization by finite element method

In this paper, the letter C (with or without subscripts) is used to denote a constant which may be different at different places. For convenience, the symbols $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$ mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$. Let $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) denote a bounded convex domain with Lipschitz boundary $\partial\Omega$. We use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see, e.g. [1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace. For simplicity, we use $\|\cdot\|_s$ to denote $\|\cdot\|_{s,2,\Omega}$ and V to denote $H_0^1(\Omega)$ in the rest of the paper.

Here, we consider the following type of semilinear elliptic equation:

$$\begin{cases} -\nabla \cdot (\mathcal{A} \nabla u) + f(x, u) &= g, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where $\mathcal{A} = (a_{i,j})_{d \times d}$ is a symmetric positive definite matrix with $a_{i,j} \in W^{1,\infty}$ ($i, j =$

$1, 2, \dots, d$), $f(x, u)$ is a nonlinear function corresponding to the second variable and satisfies the following assumption.

Assumption A: The nonlinear function $f(x, \cdot)$ has a non-negative derivative in second argument

$$0 \leq \frac{\partial f}{\partial v}(x, v) \leq C_f, \quad \forall x \in \Omega \text{ and } \forall v \in V. \quad (2.2)$$

The weak form of the semilinear problem (2.1) can be described as: Find $u \in V$ such that

$$a(u, v) + (f(x, u), v) = (g, v), \quad \forall v \in V, \quad (2.3)$$

where

$$a(u, v) = (\mathcal{A} \nabla u, \nabla v). \quad (2.4)$$

Obviously, $a(u, v)$ is bounded and coercive on V , i.e.,

$$a(u, v) \leq C_a \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \text{and} \quad c_a \|u\|_{1,\Omega}^2 \leq a(u, u), \quad \forall u, v \in V. \quad (2.5)$$

Then we use the norm $\|w\|_a := \sqrt{a(w, w)}$ for any $w \in V$ in this paper to replace the standard norm $\|\cdot\|_1$.

Now, we introduce the finite element method for semilinear elliptic problem (2.3). First we generate a shape regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$, tetrahedrons or hexahedrons for $d = 3$ (cf. [7, 8]). The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh \mathcal{T}_h , we construct the finite element space $V_h \subset V$. For simplicity, we set V_h as the linear finite element space which is defined as follows

$$V_h = \{v_h \in C(\Omega) \mid v_h|_K \in \mathcal{P}_1, \quad \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega), \quad (2.6)$$

where \mathcal{P}_1 denotes the linear function space.

The standard finite element scheme for semilinear equation (2.3) is: Find $\bar{u}_h \in V_h$ such that

$$a(\bar{u}_h, v_h) + (f(x, \bar{u}_h), v_h) = (g, v_h), \quad \forall v_h \in V_h. \quad (2.7)$$

Denote a linearized operator $L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by:

$$(Lw, v) = (\nabla w, \nabla v) + (b(x)w, v), \quad \forall w, v \in V$$

with

$$b(x) = \begin{cases} \partial f(x, u)/\partial u, & \text{if } u(x) = \bar{u}_h(x), \\ (f(x, u) - f(x, \bar{u}_h))/(u - \bar{u}_h), & \text{otherwise.} \end{cases} \quad (2.8)$$

In order to deduce the global prior error estimates, we introduce $\eta_a(V_h)$ as follows:

$$\eta_a(V_h) = \sup_{f \in L^2(\Omega), \|f\|_0=1} \inf_{v_h \in V_h} \|L^{-1}f - v_h\|_a.$$

It is easy to know that $\eta_a(V_h) \rightarrow 0$ as $h \rightarrow 0$ (cf. [7, 8]).

In order to measure the error for the finite element approximations, we denote

$$\delta_h(u) = \inf_{v_h \in V_h} \|u - v_h\|_a.$$

From [16], we can give the following error estimates.

Lemma 2.1. *When Assumption A is satisfied, equations (2.3) and (2.7) are uniquely solvable and the following estimates hold*

$$\|u - \bar{u}_h\|_a \lesssim \delta_h(u), \quad (2.9)$$

$$\|u - \bar{u}_h\|_0 \lesssim \eta_a(V_h) \|u - \bar{u}_h\|_a. \quad (2.10)$$

Furthermore, we have the following estimate

$$\|u - \bar{u}_h\|_a \leq (1 + C\eta_a(V_h))\delta_h(u). \quad (2.11)$$

Proof. The desired results (2.9)-(2.10) are the same as Lemmas 6.2.1 and 6.2.2 in [16]. Here we only give the proof for the estimate (2.11). For this aim, we define the finite element projection operator P_h by the following equation

$$a(P_h u, v_h) = a(u, v_h), \quad \forall v_h \in V_h.$$

It is easy to know that $\|u - P_h u\|_a = \delta_h(u)$. From (2.2) and (2.10), we have

$$\begin{aligned} a(P_h u - \bar{u}_h, v_h) &= a(u - \bar{u}_h, v_h) = (f(x, \bar{u}_h) - f(x, u), v_h) \\ &\leq C_f \|u - \bar{u}_h\|_0 \|v_h\|_0 \\ &\leq C\eta_a(V_h) \|u - \bar{u}_h\|_a \|v_h\|_a, \quad \forall v_h \in V_h. \end{aligned} \quad (2.12)$$

Combining (2.12) and the triangle inequality leads to the following estimates

$$\begin{aligned} \|u - \bar{u}_h\|_a &\leq \|u - P_h u\|_a + \|P_h u - \bar{u}_h\|_a \\ &\leq \delta_h(u) + C\eta_a(V_h) \|u - \bar{u}_h\|_a, \end{aligned} \quad (2.13)$$

which means that

$$\|u - \bar{u}_h\|_a \leq \frac{1}{1 - C\eta_a(V_h)} \delta_h(u) \leq (1 + C\eta_a(V_h))\delta_h(u). \quad (2.14)$$

This is the desired result (2.11) and the proof is complete. \square

3 Full multigrid method for semilinear elliptic equation

In this section, a full multigrid method for semilinear problems is proposed based on multilevel correction scheme in [17, 18]. The key point is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on an very low dimensional space. In order to carry out the multigrid method, we first generate a coarse mesh \mathcal{T}_H with the mesh size H and the linear finite element space V_H is defined on the mesh \mathcal{T}_H . Then a sequence of triangulations \mathcal{T}_{h_k} of $\Omega \subset \mathcal{R}^d$ is determined as follows. Suppose \mathcal{T}_{h_1} (produced from \mathcal{T}_H by regular refinements) is given and let \mathcal{T}_{h_k} be obtained from $\mathcal{T}_{h_{k-1}}$ via one regular refinement step (produce β^d subelements) such that

$$h_k = \frac{1}{\beta} h_{k-1}, \quad k = 2, \dots, n, \quad (3.1)$$

where the positive number β denotes the refinement index and larger than 1 (always equals 2). Based on this sequence of meshes, we construct the corresponding nested linear finite element spaces such that

$$V_H \subseteq V_{h_1} \subset V_{h_2} \subset \dots \subset V_{h_n}. \quad (3.2)$$

Since the convexity of the domain Ω , the sequence of finite element spaces $V_{h_1} \subset V_{h_2} \subset \dots \subset V_{h_n}$ and the finite element space V_H have the following relations of approximation accuracy

$$\eta_a(V_{h_k}) \approx \frac{1}{\beta} \eta_a(V_{h_{k-1}}), \quad \delta_{h_k}(u) \approx \frac{1}{\beta} \delta_{h_{k-1}}(u), \quad k = 2, \dots, n. \quad (3.3)$$

3.1 One correction step

In order to design the full multigrid method, first we introduce an one correction step in this subsection.

Assume we have obtained an approximate solution $u_{h_k}^{(\ell)} \in V_{h_k}$. The one correction step is designed as follows to improve the accuracy of the given approximation $u_{h_k}^{(\ell)}$.

Algorithm 3.1. *One Correction Step*

1. Define the following auxiliary boundary value problem: Find $\hat{u}_{h_k}^{(\ell+1)} \in V_{h_k}$ such that

$$a(\hat{u}_{h_k}^{(\ell+1)}, v_{h_k}) = -(f(x, u_{h_k}^{(\ell)}), v_{h_k}) + (g, v_{h_k}), \quad \forall v_{h_k} \in V_{h_k}. \quad (3.4)$$

Perform m multigrid iteration steps for the second order elliptic equation to obtain an approximate solution $\tilde{u}_{h_k}^{(\ell+1)}$ with the following error reduction rate

$$\|\tilde{u}_{h_k}^{(\ell+1)} - \hat{u}_{h_k}^{(\ell+1)}\|_a \leq \theta \|u_{h_k}^{(\ell)} - \hat{u}_{h_k}^{(\ell+1)}\|_a, \quad (3.5)$$

where $u_{h_k}^{(\ell)}$ is used as the initial value for the multigrid iteration and $\theta < 1$ is a fixed constant independent from the mesh size h_k .

2. Define a finite element space $V_{H,h_k} := V_H + \text{span}\{\tilde{u}_{h_k}^{\ell+1}\}$ and solve the following semilinear elliptic equation: Find $u_{h_k}^{(\ell+1)} \in V_{H,h_k}$ such that

$$a(u_{h_k}^{(\ell+1)}, v_{H,h_k}) + (f(x, u_{h_k}^{(\ell+1)}), v_{H,h_k}) = (g, v_{H,h_k}), \quad \forall v_{H,h_k} \in V_{H,h_k}. \quad (3.6)$$

In order to simplify the notation and summarize the above two steps, we define

$$u_{h_k}^{(\ell+1)} = \text{SemilinearMG}(V_H, u_{h_k}^{(\ell)}, V_{h_k}).$$

In order to give the error analysis for Algorithm 3.1, we denote a linearized operator $L_{k,\ell} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by:

$$(L_{k,\ell+1}w, v) = (\nabla w, \nabla v) + (b_{k,\ell+1}(x)w, v), \quad \forall w, v \in H_0^1(\Omega) \quad (3.7)$$

with

$$b_{k,\ell+1}(x) := \begin{cases} \partial f(x, \bar{u}_{h_k}) / \partial \bar{u}_{h_k}, & \text{if } \bar{u}_{h_k}(x) = u_{h_k}^{(\ell+1)}, \\ (f(x, \bar{u}_{h_k}) - f(x, u_{h_k}^{(\ell+1)})) / (\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}), & \text{otherwise.} \end{cases}$$

Let us introduce $\eta_{a,k,\ell+1}(V_H)$ as follows:

$$\eta_{a,k,\ell+1}(V_H) = \sup_{f \in L^2(\Omega), \|f\|_0=1} \inf_{v_H \in V_{H,h_k}} \|L_{k,\ell+1}^{-1}f - v_H\|_a. \quad (3.8)$$

From the finite element theory [7, 8] and Assumption A, it is easy to know that $\eta_{a,k,\ell+1}(V_H) \rightarrow 0$ as $H \rightarrow 0$. The error estimate of Algorithm 3.1 was studied in the following theorem.

Theorem 3.1. Assume the given solution $u_{h_k}^{(\ell)}$ has the following estimate

$$\|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_0 \lesssim \eta_{a,k,\ell}(V_H) \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_a. \quad (3.9)$$

After the one correction step defined by Algorithm 3.1, the resultant approximate solution $u_{h_k}^{(\ell+1)}$ has the following estimates

$$\|\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}\|_a \leq \gamma \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_a, \quad (3.10)$$

$$\|\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}\|_0 \leq C\eta_{a,k,\ell+1}(V_H) \|\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}\|_a, \quad (3.11)$$

where

$$\gamma := (\theta + (1 + \theta)C\eta_{a,k,\ell}(V_H))(1 + C\eta_{a,k,\ell+1}(V_H)).$$

Proof. From (2.2), (2.7) and (3.4), we have

$$\begin{aligned} a(\bar{u}_{h_k} - \hat{u}_{h_k}^{(\ell+1)}, v_{h_k}) &= (f(x, u_{h_k}^{(\ell)}) - f(x, \bar{u}_{h_k}), v_{h_k}) \\ &\leq C_f \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_0 \|v_{h_k}\|_0 \leq C_{\eta_{a,k,\ell}}(V_H) \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_a \|v_{h_k}\|_0. \end{aligned} \quad (3.12)$$

Combing (2.5) and (3.12) leads to

$$\|\bar{u}_{h_k} - \hat{u}_{h_k}^{(\ell+1)}\|_a \leq C_{\eta_{a,k,\ell}}(V_H) \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_a. \quad (3.13)$$

After performing m multigrid iteration steps, from (3.5) and (3.13), the following estimates hold

$$\begin{aligned} \|\hat{u}_{h_k}^{(\ell+1)} - \bar{u}_{h_k}\|_a &\leq \|\hat{u}_{h_k}^{(\ell+1)} - \hat{u}_{h_k}^{(\ell)}\|_a + \|\hat{u}_{h_k}^{(\ell)} - \bar{u}_{h_k}\|_a \\ &\leq \theta \|u_{h_k}^{(\ell)} - \hat{u}_{h_k}^{(\ell+1)}\|_a + \|\hat{u}_{h_k}^{(\ell+1)} - \bar{u}_{h_k}\|_a \\ &\leq \theta \|u_{h_k}^{(\ell)} - \bar{u}_{h_k}\|_a + \theta \|\hat{u}_{h_k}^{(\ell+1)} - \bar{u}_{h_k}\|_a + \|\hat{u}_{h_k}^{(\ell+1)} - \bar{u}_{h_k}\|_a \\ &\leq (\theta + (1 + \theta)C_{\eta_{a,k,\ell}}(V_H)) \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_a. \end{aligned} \quad (3.14)$$

Note that the semilinear elliptic problem (3.6) can be regarded as a finite dimensional approximation of the semilinear elliptic problem (2.7). We choose $\phi \in L^2(\Omega)$ such that $\|\phi\|_0 = 1$ and $\|\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}\|_0 = (\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}, \phi)$. Based on problems (2.7), (3.6) and the operator $L_{k,\ell+1}$ defined in (3.7), the following estimates hold

$$\begin{aligned} \|\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}\|_0 &= (\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}, \phi) = (\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}, L_{k,\ell+1}\psi) \\ &= a(\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}, \psi) + (b_{k,\ell+1}(x)(\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}), \psi) \\ &= a(\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}, \psi) + (f(\bar{u}_{h_k}) - f(u_{h_k}^{(\ell+1)}), \psi) \\ &= a(\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}, \psi - P_{H,h_k}\psi) + (f(\bar{u}_{h_k}) - f(u_{h_k}^{(\ell+1)}), \psi - P_{H,h_k}\psi), \end{aligned} \quad (3.15)$$

where $\psi \in H_0^1(\Omega)$ satisfies the equation

$$(L_{k,\ell+1}\psi, v) = (v, \phi), \quad \forall v \in H_0^1(\Omega)$$

and $P_{H,h_k} : V \rightarrow V_{H,h_k}$ denotes the finite element projection operator defined as follows

$$a(P_{H,h_k}w, v_{H,h_k}) = a(w, v_{H,h_k}), \quad \forall v_{H,h_k} \in V_{H,h_k}.$$

Combining (3.8) and (3.15) leads to the desired result (3.11). Then from the proof of Lemma 2.1, there holds

$$\begin{aligned} \|\bar{u}_{h_k} - u_{h_k}^{(\ell+1)}\|_a &\leq (1 + C_{\eta_{a,k,\ell+1}}(V_H)) \inf_{v_{H,h_k} \in V_{H,h_k}} \|\bar{u}_{h_k} - v_{H,h_k}\|_a \\ &\leq (1 + C_{\eta_{a,k,\ell+1}}(V_H)) \|\bar{u}_{h_k} - \hat{u}_{h_k}^{(\ell+1)}\|_a \\ &\leq (\theta + (1 + \theta)C_{\eta_{a,k,\ell}}(V_H)) (1 + C_{\eta_{a,k,\ell+1}}(V_H)) \|\bar{u}_{h_k} - u_{h_k}^{(\ell)}\|_a, \end{aligned}$$

which is the desired result (3.10) and the proof is complete. \square

3.2 Full multigrid method

In this subsection, a full multigrid method is proposed based on the one correction step defined in Algorithm 3.1. This algorithm can reach the optimal convergence rate with the optimal computational complexity.

Algorithm 3.2. *Full Multigrid Scheme*

1. Solve the following semilinear problem in V_{h_1} : Find $u_{h_1} \in V_{h_1}$ such that

$$a(u_{h_1}, v_{h_1}) + (f(x, u_{h_1}), v_{h_1}) = (g, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

2. For $k = 2, \dots, n$, do the following iteration:

(a) Set $u_{h_k}^{(0)} = u_{h_{k-1}}$.

(b) For $\ell = 0, \dots, p-1$, do the following iterations

$$u_{h_k}^{(\ell+1)} = \text{SemilinearMG}(V_H, u_{h_k}^{(\ell)}, V_{h_k}).$$

(c) Define $u_{h_k} = u_{h_k}^{(p)}$.

End Do

Finally, we obtain an approximate solution $u_{h_n} \in V_{h_n}$.

From the condition (2.2) and convexity of the domain Ω , the quantity $\eta_{a,k,\ell}(V_H)$ has a uniform bound. Let us define

$$\eta_a(V_H) = \max_{1 \leq k \leq n} \max_{1 \leq \ell \leq p} \eta_{a,k,\ell}(V_H). \quad (3.16)$$

From the finite element theory [7, 8], we have that $\eta_a(V_H) \rightarrow 0$ as $H \rightarrow 0$.

Theorem 3.2. *After implementing Algorithm 3.2, we have the following error estimates for the final approximation u_{h_n}*

$$\|\bar{u}_{h_n} - u_{h_n}\|_a \leq \frac{2\gamma^p \beta}{1 - \gamma^p \beta} \delta_{h_n}(u), \quad (3.17)$$

$$\|\bar{u}_{h_n} - u_{h_n}\|_0 \leq C\eta_a(V_H) \|\bar{u}_{h_n} - u_{h_n}\|_a, \quad (3.18)$$

under the condition that the coarsest mesh size H is small enough such that $\gamma^p \beta < 1$.

Proof. From the first step of Algorithm 3.2, we have $u_{h_1} = \bar{u}_{h_1}$. Then from Lemma 2.1 and the proof of Theorem 3.1, the following estimates hold

$$\begin{aligned} \|\bar{u}_{h_2} - u_{h_2}\|_a &= \|\bar{u}_{h_2} - u_{h_2}^{(p)}\|_a \leq \gamma^p \|\bar{u}_{h_2} - u_{h_2}^{(0)}\|_a \\ &= \gamma^p \|\bar{u}_{h_2} - u_{h_1}\|_a = \gamma^p \|\bar{u}_{h_2} - \bar{u}_{h_1}\|_a. \end{aligned} \quad (3.19)$$

$$\|\bar{u}_{h_2} - u_{h_2}\|_0 \leq C\eta_a(V_H)\|\bar{u}_{h_2} - u_{h_2}\|_a. \quad (3.20)$$

Based on (3.19), (3.20), Theorem 3.1 and recursive argument, the final approximate solution has the following error estimates

$$\begin{aligned} \|\bar{u}_{h_n} - u_{h_n}\|_a &\leq \gamma^p \|\bar{u}_{h_n} - u_{h_n}^{(0)}\|_a = \gamma^p \|\bar{u}_{h_n} - u_{h_{n-1}}\|_a \\ &\leq \gamma^p (\|\bar{u}_{h_n} - \bar{u}_{h_{n-1}}\|_a + \|\bar{u}_{h_{n-1}} - u_{h_{n-1}}\|_a) \\ &\leq \gamma^p \|\bar{u}_{h_n} - \bar{u}_{h_{n-1}}\|_a + \gamma^{2p} (\|\bar{u}_{h_{n-1}} - \bar{u}_{h_{n-2}}\|_a + \|\bar{u}_{h_{n-2}} - u_{h_{n-2}}\|_a) \\ &\leq \sum_{k=1}^{n-1} \gamma^{kp} \|\bar{u}_{h_{n-k+1}} - \bar{u}_{h_{n-k}}\|_a \\ &\leq \sum_{k=1}^{n-1} \gamma^{kp} (\|\bar{u}_{h_{n-k+1}} - u\|_a + \|u - \bar{u}_{h_{n-k}}\|_a) \\ &\leq 2 \sum_{k=1}^{n-1} \gamma^{kp} \delta_{h_{n-k}}(u) \leq 2 \sum_{k=1}^{n-1} \gamma^{kp} \beta^k \delta_{h_n}(u) \leq \frac{2\gamma^p \beta}{1 - \gamma^p \beta} \delta_{h_n}(u), \end{aligned}$$

which is just the the desired result (3.17). The second result (3.18) can be proved by the similar argument in the the proof of Theorem 3.1 and the proof is complete. \square

Corollary 3.1. *For the final approximation u_{h_n} obtained by Algorithm 3.2, we have the following estimates*

$$\|u - u_{h_n}\|_a \lesssim \delta_{h_n}(u), \quad (3.21)$$

$$\|u - u_{h_n}\|_0 \lesssim \eta_a(V_H) \delta_{h_n}(u). \quad (3.22)$$

Proof. This is a direct consequence of the combination of Lemma 2.1 and Theorem 3.2. \square

3.3 Estimate of the computational work

In this subsection, we turn our attention to the estimate of computational work for the full multigrid method defined in Algorithm 3.2. It will be shown that the full multigrid method makes solving the semilinear elliptic problem need almost the same work as solving the corresponding linear boundary value problems.

First, we define the dimension of each level finite element space as $N_k := \dim V_{h_k}$. Then we have

$$N_k \approx \left(\frac{1}{\beta}\right)^{d(n-k)} N_n, \quad k = 1, 2, \dots, n. \quad (3.23)$$

The computational work for the second step in Algorithm 3.2 is different from the linear elliptic problems [4, 14, 15, 16, 21]. In this step, we need to solve a semilinear elliptic problem (3.6). Always, some type of nonlinear iteration method (fixed-point

iteration or Newton type iteration) is adopted to solve this low dimensional semilinear elliptic problem. In each nonlinear iteration step, it is required to assemble the matrix on the finite element space V_{H,h_k} ($k = 2, \dots, n$) which needs the computational work $\mathcal{O}(N_k)$. Fortunately, the matrix assembling can be carried out by the parallel way easily in the finite element space since it has no data transfer.

Theorem 3.3. *Assume we use ϑ computing-nodes in Algorithm 3.2, the semilinear elliptic solving in the coarse spaces V_{H,h_k} ($k = 2, \dots, n$) and V_{h_1} need work $\mathcal{O}(M_H)$ and $\mathcal{O}(M_{h_1})$, respectively, and the work of the multigrid iteration for the boundary value problem in each level space V_{h_k} is $\mathcal{O}(N_k)$ for $k = 2, 3, \dots, n$. Let ϖ denote the nonlinear iteration times when we solve the semilinear elliptic problem (3.6). Then in each computational node, the work involved in Algorithm 3.2 has the following estimate*

$$\text{Total work} = \mathcal{O}\left(\left(1 + \frac{\varpi}{\vartheta}\right)N_n + M_H \log N_n + M_{h_1}\right). \quad (3.24)$$

Proof. We use W_k to denote the work involved in each correction step on the k -th finite element space V_{h_k} . From the definition of Algorithm 3.2, we have the following estimate

$$W_k = \mathcal{O}\left(N_k + M_H + \varpi \frac{N_k}{\vartheta}\right). \quad (3.25)$$

Based on the property (3.23), iterating (3.25) leads to

$$\begin{aligned} \text{Total work} &= \sum_{k=1}^n W_k = \mathcal{O}\left(M_{h_1} + \sum_{k=2}^n \left(N_k + M_H + \varpi \frac{N_k}{\vartheta}\right)\right) \\ &= \mathcal{O}\left(\sum_{k=2}^n \left(1 + \frac{\varpi}{\vartheta}\right)N_k + (n-1)M_H + M_{h_1}\right) \\ &= \mathcal{O}\left(\sum_{k=2}^n \left(\frac{1}{\beta}\right)^{d(n-k)} \left(1 + \frac{\varpi}{\vartheta}\right)N_n + M_H \log N_n + M_{h_1}\right) \\ &= \mathcal{O}\left(\left(1 + \frac{\varpi}{\vartheta}\right)N_n + M_H \log N_n + M_{h_1}\right). \end{aligned} \quad (3.26)$$

This is the desired result and we complete the proof. \square

Remark 3.1. *Since we always have a good enough initial solution $\tilde{u}_{h_k}^{(\ell+1)}$ in the second step of Algorithm 3.1, then solving the semilinear elliptic problem (3.6) always does not need many nonlinear iteration times. In this case, the complexity in each computational node will be $\mathcal{O}(N_n)$ provided $M_H \ll N_n$ and $M_{h_1} \leq N_n$. For more difficult nonlinear problems, the complexity in each computational node can also be bounded to $\mathcal{O}(N_n)$ by the parallel way with enough computational nodes.*

4 Numerical results

In this section, four numerical experiments are presented to verify the theoretical analysis and efficiency of Algorithm 3.2. We will check different nonlinear terms which include polynomial, exponential functions and the function only has bounded first order derivative. Furthermore, we also investigate the performance of the full multigrid method on the adaptively refined meshes. In all examples, we choose $m = 2$ and $p = 1$.

4.1 Example 1

We consider the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + u^3 = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where $\Omega = (0, 1)^3$. We choose the right hand side term g such that the exact solution is given by

$$u = \sin(\pi x) \sin(\pi y) \sin(\pi z). \quad (4.2)$$

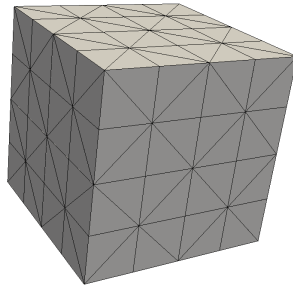


Figure 1: The initial mesh for Example 1

We give the numerical results for the approximate solutions by Algorithm 3.2. Figure 1 shows the initial triangulation. Figure 2 shows the error estimates and the CPU time in seconds. It is shown in the Figure 2 that the approximate solution by Algorithm 3.2 has the optimal convergence order and the linear computational complexity which coincides with the theoretical results in Theorems 3.1, 3.2 and Corollary 3.1.

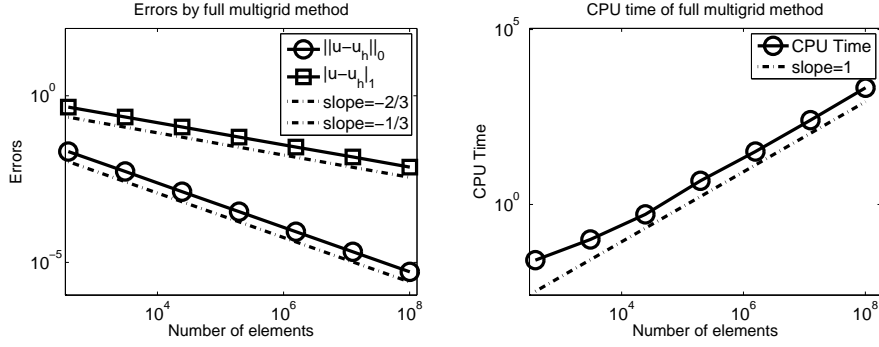


Figure 2: Errors and CPU time (in seconds) of Algorithm 3.2 for Example 1

4.2 Example 2

In the second example, we solve the following semilinear elliptic problem:

$$\begin{cases} -\Delta u - e^{-u} = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.3)$$

where $\Omega = (0,1)^3$. Since the exact solution is not known, we choose an adequate accurate approximate solution on a fine enough mesh as the exact one.

Algorithm 3.2 is applied to this example. Figure 1 shows the initial mesh. Figure 3 gives the corresponding numerical results which also show the optimal convergence rate and linear computational complexity of Algorithm 3.2.

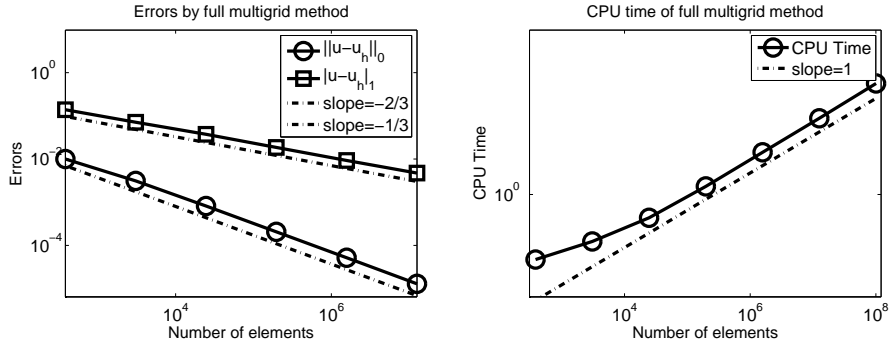


Figure 3: Errors and CPU time (in seconds) of Algorithm 3.2 for Example 2

4.3 Example 3

In the third example, we solve the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + f(x, u) = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.4)$$

with

$$f(x, u) = \begin{cases} u^{3/2}, & \text{if } u \geq 0, \\ -u^{3/2}, & \text{if } u < 0, \end{cases} \quad (4.5)$$

where $\Omega = (0, 1)^3$. We choose the right hand side term g such that the exact solution is given by

$$u = \sin(2\pi x) \sin(2\pi y) \sin(2\pi z). \quad (4.6)$$

In this example, the nonlinear term $f(x, v)$ has the bounded first order derivative $\partial f(x, v)/\partial v$ but unbounded second order derivative $\partial^2 f(x, v)/\partial^2 v$. Then the methods given in [10, 16] can not be used for this example.

Algorithm 3.2 is applied to this example. Figure 1 shows the initial mesh. Figure 4 gives the corresponding numerical results which also show the optimal convergence rate and linear computational complexity of Algorithm 3.2.

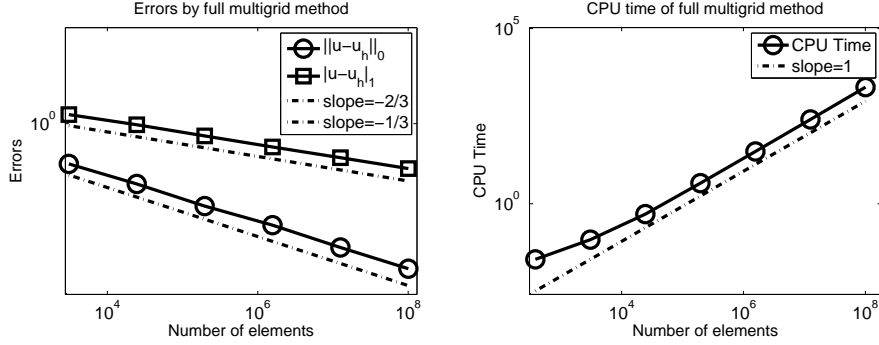


Figure 4: Errors and CPU time (in seconds) of Algorithm 3.2 for Example 3

4.4 Example 4

In the last example, we solve the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + u^{3/2} = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.7)$$

where $\Omega = (-1, 1)^3 \setminus [0, 1]^3$. Due to the reentrant corner of Ω , the exact solution with singularities is expected. The convergence order for approximate solution is less than the order predicted by the theory for regular solutions. Thus, the adaptive refinement is adopted to couple with the full multigrid method described in Algorithm 3.2 (cf. [13]).

Since the exact solution is not known, we also choose an adequately accurate approximation on a fine enough mesh as the exact one. We give the numerical

results of the full multigrid method in which the sequence of meshes $\mathcal{T}_{h_1}, \dots, \mathcal{T}_{h_n}$ is produced by the adaptive refinement with the following a posteriori error estimator

$$\eta^2(v, K) := h_K^2 \|\mathcal{R}_K(v)\|_{0,K}^2 + \sum_{e \in \mathcal{E}_I, e \subset \partial K} h_e \|\mathcal{J}_e(v)\|_{0,e}^2, \quad (4.8)$$

where the element residual $\mathcal{R}_K(v)$ and the jump residual $\mathcal{J}_e(v)$ are defined as follows:

$$\mathcal{R}_K(v) := g - f(x, v) - \nabla \cdot (\mathcal{A} \nabla v), \quad \text{in } K \in \mathcal{T}_{h_k}, \quad (4.9)$$

$$\mathcal{J}_e(v) := -\mathcal{A} \nabla v^+ \cdot \nu^+ - \mathcal{A} \nabla v^- \cdot \nu^- := [\mathcal{A} \nabla v]_e \cdot \nu_e, \quad \text{on } e \in \mathcal{E}_I. \quad (4.10)$$

Here \mathcal{E}_I denotes the set of interior faces (edges or sides) of \mathcal{T}_{h_k} and e is the common side of elements K^+ and K^- with the unit outward normals ν^+ and ν^- , respectively, and $\nu_e = \nu^-$.

Figure 5 shows the mesh after 15 times refinement and the corresponding cross section. Figure 6 shows the numerical results by Algorithm 3.2. From Figure 6, we can find that the full multigrid method can also work on the adaptive family of meshes and obtain the optimal accuracy. The full multigrid method can be coupled with the adaptive refinement naturally to produce a type of adaptive finite element method for semilinear elliptic problem where the direct nonlinear iteration in the adaptive finite element space is not required. This can also improve the overall efficiency of the adaptive finite element method for semilinear elliptic problem solving. For more information, please refer to the paper [13].

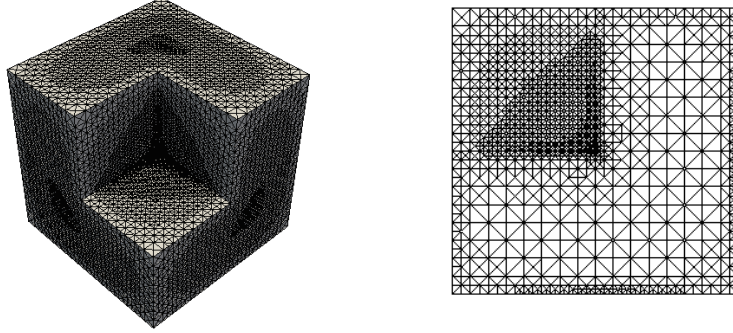


Figure 5: The triangulations after 15 adaptive refinements and the corresponding cross section for Example 4

5 Concluding remarks

In this paper, a full multigrid method is proposed for solving semilinear elliptic equations by the finite element method. The corresponding estimates of error and

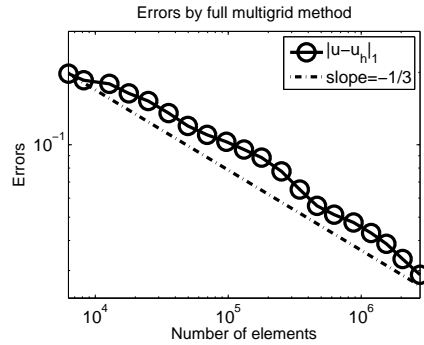


Figure 6: Errors of Algorithm 3.2 for Example 4

computational work are given. The main idea is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. Compared with the existed multigrid method which require the bounded second order derivatives of the nonlinear term, the proposed method only needs the bounded first order derivative of the nonlinear term. Based on the full multigrid method, all existed efficient solvers for the linear elliptic problems can serve as solvers for the semilinear equations.

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